

On the integral Tate conjecture for finite fields and representation theory

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ABSTRACT

We describe a new source of counterexamples to the so-called integral Hodge and integral Tate conjectures. As in the other known counterexamples to the integral Tate conjecture over finite fields, ours are approximations of the classifying space of some group BG . Unlike the other examples, we find groups of type A_n , our proof relies heavily on representation theory, and Milnor's operations vanish on the classes we construct.

1. Introduction

Recall that the original conjecture of Hodge predicted that the cycle class maps

$$\mathrm{cl}^i : \mathrm{CH}^i(X) \rightarrow \mathrm{H}^{2i}(X, \mathbb{Z})$$

would be surjective onto the subgroup of (i, i) -classes when X is a smooth projective variety. In [2] Atiyah and Hirzebruch constructed for each prime ℓ a smooth projective complex algebraic variety X and an ℓ -torsion integral cohomology class $x \in \mathrm{H}^4(X, \mathbb{Z})$, necessarily of type $(2, 2)$, such that x is not in the image of the cycle class map. They use the differentials d_r in the Atiyah-Hirzebruch spectral sequence and argue that d_r vanishes on the image of cl^i . By studying the cohomology of finite groups, specifically $(\mathbb{Z}/\ell)^3$, they exhibit degree 4 torsion classes not killed by these differentials. Using Künneth, $\mathrm{H}^*(B(\mathbb{Z}/\ell)^3, \mathbb{F}_\ell)$ has three degree 1 generators, y_1, y_2, y_3 . The class $y = \beta(y_1 y_2 y_3) \in \mathrm{H}^4(B(\mathbb{Z}/\ell)^3, \mathbb{Z})$ is the class used by Atiyah and Hirzebruch. They show that $d_{2\ell-1}(y) \neq 0$. Any smooth projective variety that looks sufficiently like $B(\mathbb{Z}/\ell)^3 \times BG_m$ then possesses a cohomology class $x \in \mathrm{H}^4(X, \mathbb{Z})$ with $d_{2\ell-1}(x) \neq 0$. Hence, this class cannot be in the image of the cycle class map. The Godeaux-Serre construction is one way to produce such varieties.

Tate's conjecture is for smooth projective varieties X defined over a field k finitely generated over its prime field and for a fixed prime ℓ different from $\mathrm{char}(k)$. It asserts that the cycle class maps in ℓ -adic cohomology

$$\mathrm{cl}^i : \mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Q}_\ell} \rightarrow \bigcup_U \mathrm{H}^{2i}(X_{\bar{k}}, \mathbb{Q}_\ell(i))^U \subseteq \mathrm{H}^{2i}(X_{\bar{k}}, \mathbb{Q}_\ell(i))$$

are surjective, where U varies over all open subgroups of $\mathrm{Gal}(\bar{k}/k)$ and \bar{k} is the algebraic closure of k .

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There is an integral version of Tate's conjecture in which $\mathbb{Q}_\ell(i)$ is replaced by $\mathbb{Z}_\ell(i)$, but, like the integral Hodge conjecture, it is false and it appears that it was not expected by Tate. Colliot-Thélène and Szamuely adapted the examples of Atiyah and Hirzebruch to this situation in [5, Théorème 2.1]. Non-torsion classes not in the image of the cycle class map (either in singular cohomology over \mathbb{C} or in ℓ -adic cohomology) were constructed over uncountable fields by Kollár using very general hypersurface high-degree hypersurfaces in $\mathbb{P}_{\mathbb{C}}^4$.

Since the examples mentioned so far use either torsion cohomology classes or an uncountable ground field, Colliot-Thélène and Szamuely asked in [5, Remark 2.2.3] whether a modified integral Tate conjecture asserting the surjectivity of

$$\mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Z}_\ell} \rightarrow \mathrm{H}^{2i}(X_{\bar{k}}, \mathbb{Z}_\ell(i)) / \text{torsion} \quad (1)$$

could still be true for smooth projective varieties over finite fields.

However, in [12], Pirutka and Yagita constructed counterexamples to the surjectivity of (1) over finite fields for $\ell = 2, 3, 5$. Their examples are smooth projective approximations to the classifying spaces $B(G \times G_m)$ in characteristic p for G a simply connected affine algebraic group in the following cases: G_2 at the prime $\ell = 2$, F_4 at the prime $\ell = 3$, and E_8 at the prime $\ell = 5$. Each of these groups has $\mathrm{H}^4(BG, \mathbb{Z}) = \mathbb{Z} \cdot x$ for some generator x , and they show that x is not in the image of the cycle class map. This result gives new counterexamples to the integral Hodge conjecture and, by using integral models and reduction modulo p , gives counterexamples to the modulo-torsion integral Tate conjecture over finite fields, answering the question of Colliot-Thélène and Szamuely.

Shortly after the appearance of [12], Kameko [10] gave examples for all primes ℓ by studying $B(\mathrm{SL}_\ell \times \mathrm{SL}_\ell / \mu_\ell)$, where μ_ℓ is embedded diagonally.

In all of these examples over finite fields, the argument actually goes by embedding the disruptive group $(\mathbb{Z}/\ell)^3$ into G and then showing that some $x \in \mathrm{H}^4(BG, \mathbb{Z})$ restricts to the class $y \in \mathrm{H}^4(B(\mathbb{Z}/\ell)^3, \mathbb{Z})$ found by Atiyah and Hirzebruch. In other words, they show that $d_{2\ell-1}(x) \neq 0$ in the Atiyah-Hirzebruch spectral sequence.

The purpose of the present paper is to return to these questions and give a different type of argument, for certain groups of type A_n , relying on representation theory.

Let G be a quotient of SL_n by a central subgroup $C \subseteq \mu_n$. Then, $\mathrm{H}^4(BG, \mathbb{Z}) \rightarrow \mathrm{H}^4(B\mathrm{SL}_n, \mathbb{Z})$ is injective, and we can write $\mathrm{H}^4(BG, \mathbb{Z})$ as $\mathbb{Z} \cdot nc_2(\gamma_n)$, some multiple of c_2 of the standard n -dimensional representation of SL_n . Note however, that unless C is trivial γ_n does not descent to a representation of G .

THEOREM 1.1. *Let G be either SL_8 / μ_2 or SL_9 / μ_3 , viewed as complex affine algebraic groups. Then, $\mathrm{H}^4(BG, \mathbb{Z}) = \mathbb{Z} \cdot c_2$, while the image of $\mathrm{CH}^2(BG) \rightarrow \mathrm{H}^4(BG, \mathbb{Z})$ is $2c_2 \cdot \mathbb{Z}$ (resp. $3c_2 \cdot \mathbb{Z}$).*

Standard arguments then give counterexamples to the mod-torsion Hodge and Tate conjectures in characteristic zero and the Tate conjecture over \mathbb{F}_p for $\ell = 2, 3$ for all but finitely many primes p . In particular, we obtain a new proof of the following theorem of Pirutka-Yagita (for $\ell = 2, 3, 5$) and Kameko (for all ℓ).

THEOREM 1.2 [12, Theorem 1.1], [10, Theorem 1.1]. *Let $\ell = 2, 3$. Then, for all primes $p \neq \ell$ there exists a smooth projective variety X over \mathbb{F}_p such that the cycle map*

$$\mathrm{cl}^2 : \mathrm{CH}^2(X_{\bar{k}})_{\mathbb{Z}_\ell} \rightarrow \bigcup_U \mathrm{H}^4(X_{\bar{k}}, \mathbb{Z}_\ell(i))^U / \text{torsion}$$

is not surjective.

Our arguments not only differ from those used by Pirutka, Yagita, and Kameko, but they must differ. Indeed, we show that in the cases of Theorem 1.1 the differential $d_{2\ell-1}(c_2) = 0$ in contrast to the work of the previous authors. Moreover, if $\rho(c_2)$ is the reduction modulo ℓ of $c_2 \in H^4(BG, \mathbb{Z})$ for G as in the theorem, then all of Milnor's operations Q_i vanish on $\rho(c_2)$. However, thanks to the Atiyah-Segal completion theorem, some later differential d_r in the spectral sequence does detect the non-algebraicity of c_2 ; see Proposition 2.5.

The approach we discovered is checkable for $\ell = 2, 3$ and in theory for SL_{ℓ^2}/μ_{ℓ} for any given odd prime ℓ . Namely, we use the fact that $CH^2(BG)$ is generated by the Chern classes of polynomial representations of G . We then determine generators for the representation rings of SL_8/μ_2 and SL_9/μ_3 and compute, using a short SAGE [14] program, the second Chern classes of all of these representations. The theorem follows immediately from this computation.

CONJECTURE 1.3. *If ℓ is an odd prime, the image of $CH^2(BG) \rightarrow H^4(BG, \mathbb{Z}) = \mathbb{Z} \cdot c_2$ is generated by ℓc_2 for $G = SL_{\ell^2}/\mu_{\ell}$.*

This paper is organized as follows. In Section 2 we prove Theorem 1.1. In Section 3 we construct smooth projective examples and reduce modulo p . Finally, in Section 4 we explain the SAGE code we use for computations.

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2. Examples in characteristic zero

In this section we work with affine algebraic groups G over the complex numbers. Implicitly, by $H^*(BG, \mathbb{Z})$ we mean $H^*(BG(\mathbb{C}), \mathbb{Z})$ and so on. We will use the following result from Totaro's book, which can also be found in the work of Esnault-Kahn-Levine-Viehweg.

THEOREM 2.1 [17, Theorem 2.25], [7, Lemma C.3]. *If G is an affine algebraic group scheme over a field k , then $CH^2(BG)$ is generated by Chern classes of polynomial representation.*

Note that the statement in the book is for modulo ℓ Chow groups, but that the proof works integrally for $CH^2(BG)$.

We say that the integral Hodge conjecture holds in codimension 2 for BG where G is an affine algebraic group over \mathbb{C} if $H^4(BG, \mathbb{Z})$ is generated by Chern classes of representations. This is justified by Theorem 2.1 and the fact that when it fails one can construct counterexamples to the integral Hodge conjecture in codimension 2 on smooth projective complex varieties.

2.1 Groups of type A_n for small n

It is well-known that for $G = SL_n$ or $G = GL_n$ the cycle class maps give ring isomorphisms $CH^*(BG) \rightarrow H^*(BG, \mathbb{Z})$. See [17, Chapter 2]. In other low-dimensional cases, we can obtain a similar conclusion, at least for $H^4(BG, \mathbb{Z})$. Write γ_n for the standard n -dimensional representation of SL_n .

PROPOSITION 2.2. *For $G = \mathrm{PGL}_2, \mathrm{PGL}_3, \mathrm{SL}_4 / \mu_2, \mathrm{PGL}_4, \mathrm{PGL}_5, \mathrm{SL}_6 / \mu_2, \mathrm{SL}_6 / \mu_3, \mathrm{PGL}_6$, or PGL_7 , $H^4(\mathrm{BG}, \mathbb{Z})$ is generated by Chern classes of polynomial representations.*

Proof. For PGL_3 , PGL_5 , and PGL_7 , this follows from the work of Vistoli [19], and also from Vezzosi [18] for PGL_3 . Indeed, Vistoli shows more generally that $\mathrm{CH}^*(\mathrm{BPGL}_p) \rightarrow \mathrm{H}^*(\mathrm{BPGL}_p, \mathbb{Z})$ is an isomorphism onto the even cohomology for p an odd prime. The case of $\mathrm{PGL}_2 \cong \mathrm{SO}_3$ follows from Pandharipande’s calculations in [11]. The claim now follows from Theorem 2.1.

The group $H^4(\mathrm{BPGL}_n, \mathbb{Z}) \subseteq H^4(\mathrm{BSL}_n, \mathbb{Z}) = \mathbb{Z} \cdot c_2$ is generated by $2nc_2$ for n even and nc_2 for n odd. On the other hand, the adjoint representation has $c_2(\mathrm{Ad}_{\mathrm{PGL}_n}) = 2nc_2$. For details, see [21, Sections 1,2]. For n even it then follows that $H^4(\mathrm{BPGL}_n, \mathbb{Z})$ is generated by Chern classes, so this takes care of PGL_4 and PGL_6 .

One checks, as we will do in detail for SL_8/μ_2 in the next section, that in the remaining cases we have

$$H^4(\mathrm{BSL}_4 / \mu_2, \mathbb{Z}) = \mathbb{Z} \cdot 2c_2,$$

$$H^4(\mathrm{BSL}_6 / \mu_2, \mathbb{Z}) = \mathbb{Z} \cdot 4c_2,$$

$$H^4(\mathrm{BSL}_6 / \mu_3, \mathbb{Z}) = \mathbb{Z} \cdot 3c_2.$$

In these cases, $c_2(\wedge^2 \gamma_4) = 2c_2$, $c_2(\wedge^2 \gamma_6) = 4c_2$, $c_2(\wedge^3 \gamma_6) = 6c_2$, and $c_2(\gamma_6^{(2,1)}) = 33c_2$, where $\gamma_6^{(2,1)}$ is the irreducible representation of SL_6 corresponding to the partition $(2, 1)$. See Section 4 for details. In particular, we see that Chern classes generate $H^4(\mathrm{BG}, \mathbb{Z})$ in each of the remaining cases. These representations all descend to the quotient groups because they are trivial on the given central subgroups. \square

2.2 BSL₈ / μ_2

Let $G = \mathrm{SL}_8 / \mu_2$ over \mathbb{C} .

THEOREM 2.3. *The pullback map $H^4(\mathrm{BG}, \mathbb{Z}) \rightarrow H^4(\mathrm{BSL}_8, \mathbb{Z}) = \mathbb{Z} \cdot c_2$ is an isomorphism, while the image of the cycle class map cl^2 is $\mathbb{Z} \cdot 2c_2$ in $H^4(\mathrm{BG}, \mathbb{Z})$. In particular, the integral Hodge conjecture fails in codimension 2 for $\mathrm{BG} = \mathrm{BSL}_8 / \mu_2$.*

Proof. Consider the E_2 -page of the Leray-Serre spectral sequence for the fibration $B\mu_2 \rightarrow BSL_8 \rightarrow BG$ displayed in Figure 1.

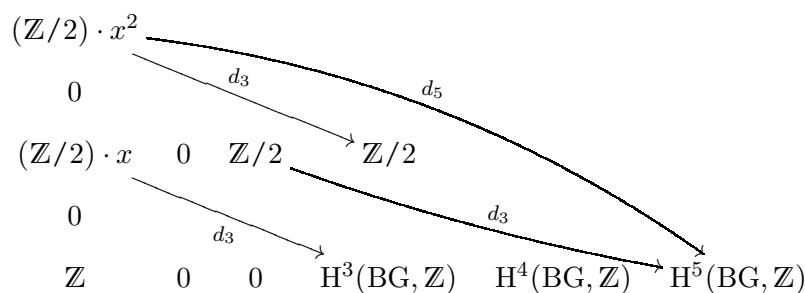


FIGURE 1. The E_2 -page of the Leray-Serre spectral sequence associated with $B\mu_2 \rightarrow BSL_8 \rightarrow BSL_8/\mu_2$.

The differential $d_3^{0,2}$ is an isomorphism as $H^3(BG, \mathbb{Z}) \cong \mathbb{Z}/2$, and we know that $H^*(BSL_8, \mathbb{Z})$ is torsion-free. Using the Leibniz rule, we see that $d_3(x^2) = 0$. Moreover, $H^5(BG, \mathbb{Z}) \cong \mathbb{Z}/4$ by

the proof of [1, Proposition 6.2]. It follows that $d_3^{2,2}$ is non-zero, as is $d_5^{4,0}$, for otherwise there would be torsion in $H^5(\mathrm{BSL}_8, \mathbb{Z})$. This shows that the edge map $H^4(\mathrm{BG}, \mathbb{Z}) \rightarrow H^4(\mathrm{BSL}_8, \mathbb{Z})$ is an isomorphism.

It follows that given a cycle $x \in \mathrm{CH}^2(\mathrm{BG})$, we can write $\mathrm{cl}^2(x) = n_x c_2$, where $c_2 \in H^4(\mathrm{BSL}_8, \mathbb{Z})$ is the Chern class of the standard representation γ_8 of SL_8 . We will see that n_x is always even.

By Theorem 2.1, to finish the proof it is enough to compute the second Chern classes of generating representations of the representation ring $R[G]$. Let L_1, \dots, L_8 generate the weight lattice of SL_8 , with fundamental weights $\alpha_i = L_1 + \dots + L_i$ for $1 \leq i \leq 7$. An irreducible representation with highest weight vector $\lambda_1 L_1 + \dots + \lambda_8 L_8$ of SL_8 descends to a representation of G if and only if $\sum_i \lambda_i$ is even. Table 1 gives the generators of the weight lattice of G , the associated Young diagrams, and the numbers n_λ such that $c_2(\gamma_8^\lambda) = n_\lambda c_2(\gamma_8) = n_\lambda c_2$ in $H^4(\mathrm{BSL}_8, \mathbb{Z})$. See Section 4 for how the table was computed. By inspection, these Chern classes are all even with greatest common divisor 2, from which the theorem follows. \square

Weight	λ	n_λ	Weight	λ	n_λ
$2\alpha_1$	(2)	16	$\alpha_1 + \alpha_3$	(2, 1, 1)	156
α_2	(1, 1)	6	$\alpha_1 + \alpha_5$	(2, 1, 1, 1, 1)	170
$2\alpha_3$	(2, 2, 2)	700	$\alpha_1 + \alpha_7$	(2, 1, 1, 1, 1, 1, 1)	16
α_4	(1, 1, 1, 1)	20	$\alpha_3 + \alpha_5$	(2, 2, 2, 1, 1)	1344
$2\alpha_5$	(2, 2, 2, 2, 2)	700	$\alpha_3 + \alpha_7$	(2, 2, 2, 1, 1, 1, 1)	170
α_6	(1, 1, 1, 1, 1, 1)	6	$\alpha_5 + \alpha_7$	(2, 2, 2, 2, 2, 1, 1)	156
$2\alpha_7$	(2, 2, 2, 2, 2, 2, 2)	10			

TABLE 1. The second Chern classes of generators of $R[\mathrm{SL}_8 / \mu_2]$.

The class $c_2 \in H^4(\mathrm{BG}, \mathbb{Z})$ is a non-torsion counterexample to the integral Hodge conjecture, which we see directly from representation theory. However, it is nevertheless important to know that some cohomology operation tells us that c_2 cannot be a cycle class. This is critical when we want to pass from BG to a smooth projective variety, either over \mathbb{C} or over a finite field.

In all previous group-theoretic counterexamples at the prime $\ell = 2$, such as those of Atiyah-Hirzebruch, Totaro, Pirutka-Yagita, or Kameko, one has $d_3(x) = \mathrm{Sq}_{\mathbb{Z}}^3(x) \neq 0$, where $d_3 = \mathrm{Sq}_{\mathbb{Z}}^3$ is the differential in the Atiyah-Hirzebruch spectral sequence converging to $\mathrm{KU}^*(\mathrm{BG})$, where KU is complex topological K -theory. This is proved by showing more strongly that the Milnor operation Q_1 does not vanish on $\rho(x)$. In our case, we have $d_3(c_2) = 0$, and moreover $Q_i(\rho(c_2)) = 0$ for all Milnor operations Q_i , where $\rho : H^*(\mathrm{BG}, \mathbb{Z}) \rightarrow H^*(\mathrm{BG}, \mathbb{Z}/2)$.

Recall that the Milnor operations are stable cohomology operations

$$Q_i : H^m(X, \mathbb{Z}/\ell) \rightarrow H^{m+2\ell^i-1}(X, \mathbb{Z}/\ell).$$

These were defined in motivic cohomology as well by Voevodsky [20, Section 9], where we refer the reader for an overview. The key point is that Q_i vanishes on the image of the cycle class map.

THEOREM 2.4. *Let $c_2 \in H^4(\mathrm{BG}, \mathbb{Z})$ be the generator, where $G = \mathrm{SL}_8 / \mu_2$, and let $x = \rho(c_2)$. Then, $Q_i(x) = 0$ for all $i \geq 0$.*

Proof. The operation Q_0 is the Bockstein operation, so it follows immediately that $Q_0(x) = 0$ since x is the reduction of an integral class. The argument that $Q_1(x) = 0$ is more subtle, as it involves the Postnikov tower for BG. Consider the commutative diagram of Postnikov sections:

$$\begin{array}{ccccccc} K(\mathbb{Z}, 6) & \longrightarrow & \tau_{\leq 6} \text{BSL}_8 & \longrightarrow & K(\mathbb{Z}, 4) & \xrightarrow{k_6} & K(\mathbb{Z}, 7) \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\ K(\mathbb{Z}, 6) & \longrightarrow & \tau_{\leq 6} \text{BG} & \longrightarrow & K(\mathbb{Z}/2) \times K(\mathbb{Z}, 4) & \longrightarrow & K(\mathbb{Z}, 7). \end{array}$$

Recall that for a connected topological space X and $n \geq 0$, the Postnikov section $\tau_{\leq n} X$ is a space with $\pi_i \tau_{\leq n} X = 0$ for $i > n$ equipped with an $n+1$ -equivalence $X \rightarrow \tau_{\leq n} X$. Since $H^7(K(\mathbb{Z}, 4), \mathbb{Z}) = \mathbb{Z}/2$, and because the cohomology of BSL_8 is non-torsion, we see that k_6 must exactly be the non-zero cohomology class. But, this class is exactly the integral operation $\text{Sq}_{\mathbb{Z}}^3$ of the fundamental class of $K(\mathbb{Z}, 4)$. See [2, Section 7] for the relationship between the operation $\text{Sq}_{\mathbb{Z}}^3$ and d_3 . It follows from the commutative diagram that $d_3(c_2) = \text{Sq}_{\mathbb{Z}}^3(c_2) = 0$ since we construct $\tau_{\leq 6} \text{BG}$ exactly using this class as the k -invariant. Now, $Q_1(x) = Q_1(\rho(c_2)) = \rho(\text{Sq}_{\mathbb{Z}}^3(c_2)) = 0$.

In general, one has $Q_i = [P^{\ell^{i-1}}, Q_{i-1}]$, the commutator in the Steenrod algebra. The cohomology operations P^i vanish on $H^4(X, \mathbb{Z}/2)$ for $i > 2$ for degree reasons for any space X . It follows it is enough to show that $Q_2 = 0$ because after that all higher Milnor operations will vanish for degree reasons on P^i and the commutator relation. Since $Q_2 = [P^1, Q_1] = [\text{Sq}^2, Q_1]$ at the prime 2, we have $Q_2(x) = Q_1(\text{Sq}^2(x))$. Now, $\text{Sq}^2(x) = x^2$ since x has degree 4, and we can use that $Q_1 = [P^1, Q_0]$ to see that

$$Q_2(x) = Q_1(\text{Sq}^2(x)) = Q_1(x^2) = Q_0(P^1(x^2)) = Q_0(\text{Sq}^1(x) \text{Sq}^1(x))$$

by the Cartan formula and the fact that $Q_0(x^2) = 0$ since x is the reduction of an integral class. For the same reason, $\text{Sq}^1(x) = 0$, and hence $P^2(x) = 0$. All notation and relations are as in [20]. \square

We need to know that nevertheless some cohomology operation detects the non-algebraicity of c_2 . This is provided by the next proposition.

PROPOSITION 2.5. *For some odd integer $r > 3$, the differential $d_r(c_2) \neq 0$ in the Atiyah-Hirzebruch sequence for $\text{BG} = \text{BSL}_8 / \mu_2$.*

Proof. We use the Atiyah-Segal completion theorem [3], which says that $\text{KU}^*(\text{BG})$ is isomorphic to the completion $\text{R}[G]_{\hat{I}}$ of $\text{R}[G]$ at the augmentation ideal, where KU denotes complex topological K -theory. Strictly speaking, the completion theorem is about compact Lie groups, but in this case there is no difference either in terms of representation theory or of K -theory in working with $G = \text{SL}_8 / \mu_2$ versus SU_8 / μ_2 . Both $\text{R}[G]_{\hat{I}}$ and $\text{KU}^0(\text{BG})$ are natural filtered, the former by the powers of the augmentation ideal and the latter as the abutment the Atiyah-Hirzebruch spectral sequence. The Atiyah-Segal isomorphism is compatible with this filtration, which means in particular that the image of $\text{KU}^*(\text{BG}) \rightarrow H^4(\text{BG}, \mathbb{Z})$ is the same as the image of $c_2 : I^2 / I^3$, since $H^2(\text{BG}, \mathbb{Z}) = 0$. But, we have already found that this map is not surjective, so that it follows that c_2 must not be permanent in the Atiyah-Hirzebruch spectral sequence. The proposition follows. \square

2.3 BSL_9 / μ_3

We take a similar approach for $G = \text{BSL}_9 / \mu_3$ over \mathbb{C} .

THEOREM 2.6. *The pullback map $H^4(BG, \mathbb{Z}) \rightarrow H^4(BSL_9, \mathbb{Z}) = \mathbb{Z} \cdot c_2$ is an isomorphism, while the image of the cycle class map cl^2 is $\mathbb{Z} \cdot 3c_2$ in $H^4(BG, \mathbb{Z})$. In particular, the integral Hodge conjecture fails in codimension 2 for $BG = BSL_9 / \mu_3$.*

PROPOSITION 2.7. *For any odd prime ℓ , the pullback map $H^4(BG, \mathbb{Z}) \rightarrow H^4(BSL_{\ell^2}, \mathbb{Z}) = \mathbb{Z} \cdot c_2$ is an isomorphism.*

Proof. As above, we consider the E_2 -page of the Leray-Serre spectral sequence $B\mu_\ell \rightarrow BSL_{\ell^2} \rightarrow BG$ displayed in Figure 2.

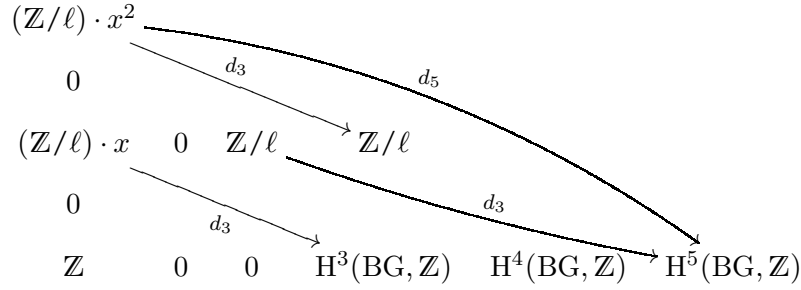


FIGURE 2. The E_2 -page of the Leray-Serre spectral sequence associated with $B\mu_\ell \rightarrow BSL_{\ell^2} \rightarrow BSL_{\ell^2} / \mu_\ell$.

The main outside input this time from [1, Proposition 6.2] is that $H^5(BG, \mathbb{Z}) \cong \mathbb{Z}/\ell$. However, in this case, $d_3^{0,2}$ is an isomorphism as is $d_3^{0,4}$, by the Leibniz rule. It follows that $d_3^{2,2}$ is also an isomorphism, from which it follows that the edge map $H^4(BG, \mathbb{Z}) \rightarrow H^4(BSL_9, \mathbb{Z})$ is an isomorphism. \square

Proof of Theorem 2.6. As above, it suffices to compute the Chern classes of generators of the representation ring of $G = SL_9 / \mu_3$. This time there are 23 generators. The results are displayed in Table 2. The theorem follows by inspection. \square

Weight	λ	n_λ	Weight	λ	n_λ
$3\alpha_1$	(3)	165	$2\alpha_1 + \alpha_7$	(3, 1, 1, 1, 1, 1, 1)	693
$3\alpha_2$	(3, 3)	3465	$\alpha_2 + \alpha_4$	(2, 2, 1, 1)	1701
α_3	(1, 1, 1)	21	$\alpha_2 + \alpha_7$	(2, 2, 1, 1, 1, 1, 1)	486
$3\alpha_4$	(3, 3, 3, 3)	116424	$2\alpha_2 + \alpha_5$	(3, 3, 1, 1, 1)	37125
$3\alpha_5$	(3, 3, 3, 3, 3)	116424	$2\alpha_2 + \alpha_8$	(3, 3, 1, 1, 1, 1, 1, 1)	2541
α_6	(1, 1, 1, 1, 1, 1)	21	$\alpha_4 + \alpha_5$	(2, 2, 2, 2, 1)	5292
$3\alpha_7$	(3, 3, 3, 3, 3, 3, 3)	3465	$\alpha_4 + \alpha_8$	(2, 2, 2, 2, 1, 1, 1, 1)	420
$3\alpha_8$	(3, 3, 3, 3, 3, 3, 3, 3)	66	$2\alpha_4 + \alpha_7$	(3, 3, 3, 3, 1, 1, 1)	117810
$\alpha_1 + \alpha_2$	(2, 1)	78	$\alpha_5 + \alpha_7$	(2, 2, 2, 2, 2, 1, 1)	1701
$\alpha_1 + \alpha_5$	(2, 1, 1, 1, 1)	420	$2\alpha_5 + \alpha_8$	(3, 3, 3, 3, 3, 1, 1, 1)	29106
$\alpha_1 + \alpha_8$	(2, 1, 1, 1, 1, 1, 1, 1)	18	$\alpha_7 + \alpha_8$	(2, 2, 2, 2, 2, 2, 2, 1)	78
$2\alpha_1 + \alpha_4$	(3, 1, 1, 1)	2541			

TABLE 2. The second Chern classes of generators of $R[SL_9 / \mu_3]$.

We include without proof the following results, as the proofs are minor modifications of the proofs of Theorem 2.4 and Proposition 2.5.

THEOREM 2.8. *Let $c_2 \in H^4(BG, \mathbb{Z})$ be the generator, where $G = \mathrm{SL}_9 / \mu_3$, and let $x = \rho(c_2) \in H^4(BG, \mathbb{Z}/3)$. Then, $Q_i(x) = 0$ for all $i \geq 0$.*

PROPOSITION 2.9. *For some odd integer $r > 5$, $d_r(c_2) \neq 0$ in the cohomology of $BG = \mathrm{BSL}_9 / \mu_3$.*

2.4 $\mathrm{BSL}_{\ell^2} / \mu_\ell$

It is clear that we cannot simply go on computing representations to get a general result. Nevertheless, for odd primes $\ell > 3$ we expect by Proposition 2.7 that a result similar to Theorem 2.6 holds.

CONJECTURE 2.10. *Fix an odd prime ℓ . Let $G = \mathrm{SL}_{\ell^2} / \mu_\ell$ over \mathbb{C} . Then, the image of the cycle class map cl^2 is the subgroup $\mathbb{Z} \cdot \ell c_2 \subseteq \mathbb{Z} \cdot c_2 = H^4(BG, \mathbb{Z})$.*

We intend to return to this question in future work with Ben Williams on the groups $\mathrm{BSL}_{\ell^2} / \mu_\ell$ and the topological period-index problem.

3. Counterexamples to the integral Hodge and Tate conjectures

There is a standard way to pass from counterexamples to the integral Hodge conjecture in the cohomology of BG to actual counterexamples in the cohomology of smooth projective complex varieties. There is also a method for defining these integrally and reducing modulo p to obtain characteristic p counterexamples to the integral Tate conjecture.

We give two modifications to these methods. The first is that we use Poonen's work on Bertini theorems over finite fields [13] to obtain results over *every* finite field different from ℓ . The second is that we make an argument using Thomason's paper on étale cohomology [15] to show that certain cohomology operations in singular cohomology with \mathbb{Z}_ℓ -coefficients come from functorial cohomology operations in ℓ -adic cohomology. This step is necessary to ensure that upon reducing to finite characteristic we still have access to an operation that detects the non-algebraicity of the class c_2 .

THEOREM 3.1. *Fix a prime $\ell = 2, 3$. Let $BG = \mathrm{BSL}_8 / \mu_2$ or BSL_9 / μ_3 according to whether $\ell = 2$ or $\ell = 3$. There exists a smooth projective complex algebraic variety X and a map $f : X \rightarrow BG$ such that $x = f^*(c_2)$ is a non-torsion Hodge class in $H^4(X, \mathbb{Z})$ that is not a cycle class, while ℓx is a cycle class. Moreover, $d_{2\ell-1}(x) = 0$.*

Proof. We leave the proof to the reader as an application of Theorems 2.4 and Theorem 2.8 and an easy modification of the proof of the next theorem. \square

The reader should compare the theorem to the original result of Atiyah and Hirzebruch where the non-algebraicity of x is exactly detected by the fact that $d_{2\ell-1}(x) \neq 0$.

THEOREM 3.2 [12, Theorem 1.1], [10, Theorem 1.1]. *Let $\ell = 2, 3$. Then, for all primes $p \neq \ell$ there exists a smooth projective variety X over \mathbb{F}_p such that the cycle class map*

$$\mathrm{cl}^2 : \mathrm{CH}^2(X_{\bar{k}})_{\mathbb{Z}_\ell} \rightarrow \bigcup_U H^4(X_{\bar{k}}, \mathbb{Z}_\ell(i))^U / \text{torsion}$$

is not surjective, where U varies over the open subgroups of $\mathrm{Gal}(\bar{k}/k)$. Moreover, the Milnor operations Q_i vanish on $H^4(X, \mathbb{Z}/\ell(2))$.

Proof. Let $G = \mathrm{SL}_{8,\mathbb{C}}/\mu_2$ or $G = \mathrm{SL}_{9,\mathbb{C}}/\mu_3$, according to the choice of ℓ . Let $r > 3$ be the unique (necessarily odd) integer such that $d_r(c_2) \neq 0$ in $H^*(\mathrm{BG}, \mathbb{Z})$, where d_r is the differential in the Atiyah-Hirzebruch spectral sequence computing $\mathrm{KU}^*(\mathrm{BG})$.

Now, let $G_{\mathbb{Z}[1/\ell]}$ denote either $\mathrm{SL}_{8,\mathbb{Z}[1/2]}/\mu_2$ or $\mathrm{SL}_{9,\mathbb{Z}[1/3]}/\mu_3$, smooth group-schemes over appropriate localizations of \mathbb{Z} . The arguments in [12, Section 4] show how to construct a smooth quasi-projective t -dimensional scheme $Y \subseteq \mathbb{P}_{\mathbb{Z}[1/\ell]}^N$ over $\mathrm{Spec} \mathbb{Z}[1/\ell]$ such that there is a map $Y \rightarrow \mathrm{BG}_{m,\mathbb{Z}[1/\ell]} \times \mathrm{BG}_{\mathbb{Z}[1/\ell]}$ having the following properties:

- (i) the induced map $Y_{\mathbb{C}} \rightarrow \mathrm{BG}_{m,\mathbb{C}} \times \mathrm{BG}_{\mathbb{C}}$ is an $r + 4$ -equivalence;
- (ii) the complement W of Y in the closure of Y in $\mathbb{P}_{\mathbb{Z}[1/\ell]}^N$ has large codimension c (to be specified below).

Let $s = \mathrm{Spec} \mathbb{F}_p$ be a closed point of $\mathrm{Spec} \mathbb{Z}[1/\ell]$ (so $p \neq \ell$) and \bar{s} the associated geometric point. Write Y_s for the closed fiber of Y over s .

We claim that there exists a complete intersection $V_s \subseteq \mathbb{P}_{\mathbb{F}_p}^N$ of codimension $t - c + 1$ such that $V_s \cap Y_s$ is a smooth *projective* variety. It suffices to show that there is a hypersurface H_s such that $H_s \cap Y_s$ is smooth and $\dim H_s \cap W_s = \dim W_s - 1$, for then we can simply continue to cut down with hypersurfaces until the boundary is empty. We use Poonen's Bertini theorem with Taylor coefficients [13, Theorem 1.2]. Indeed, if we let $T_s \subseteq W_s$ be a finite subscheme containing at least one closed point of each irreducible geometric component of W_s , then Poonen's theorem implies that there is a positive density of hypersurfaces $H_s : f = 0$ such that $H_s \cap Y_s$ is smooth of dimension $\dim Y_s - 1$ and f does not vanish at any point of T_s . In particular, it follows that $W_s \cap H_s$ must have dimension (at most) 1 less than the dimension of W_s , as desired. Iterating this argument, we find V_s .

Let us prove that $X_s = V_s \cap Y_s$ has the required properties. Since V_s is a complete intersection, it lifts to characteristic zero, i.e., to flat scheme V over the p -adics \mathbb{Z}_p . Moreover, $X = V \cap Y_{\mathbb{Z}_p}$ is a smooth projective scheme over the p -adic integers. By Hamm's quasi-projective Lefschetz hyperplane theorem [9, Section II.1.2],

$$H_{\mathrm{ét}}^i(\mathrm{BG}_{m,\mathbb{C}} \times \mathrm{BG}_{\mathbb{C}}, \mathbb{Z}_{\ell}(j)) \cong H_{\mathrm{ét}}^i(Y_{\mathbb{C}}, \mathbb{Z}_{\ell}(j)) \cong H_{\mathrm{ét}}^i(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(j))$$

for $i \leq \dim(X) - 2 = (c - 1) - 2 = c - 3$. By choosing Y so that $r + 4 \leq c - 3$ (or $r + 7 \leq c$), one finds that the inequality will hold for $i \leq r + 4$. That it is possible to find such a Y is proven in [16, Remark 1.4].

Using the invariance of étale cohomology for extensions of algebraically closed fields, and proper base change, there are maps

$$H_{\mathrm{ét}}^i(\mathrm{BG}_{m,\mathbb{C}} \times \mathrm{BG}_{\mathbb{C}}, \mathbb{Z}_{\ell}(j)) \cong H_{\mathrm{ét}}^i(Y_{\mathbb{C}}, \mathbb{Z}_{\ell}(j)) \cong H^i(X_{\mathbb{C}}, \mathbb{Z}_{\ell}(i)) \rightarrow H_{\mathrm{ét}}^i(X_{\bar{s}}, \mathbb{Z}_{\ell}(j))$$

which are isomorphisms for $i \leq r + 4$. Using the comparison isomorphism between ℓ -adic cohomology and singular cohomology for smooth complex schemes, we see that there is a class $c_2 \in H_{\mathrm{ét}}^4(X_{\bar{s}}, \mathbb{Z}_{\ell}(2))$ corresponding to $c_2 \in H^4(\mathrm{BG}, \mathbb{Z}_{\ell})$. Note that by looking at the Chern classes of representations of G we see that ℓc_2 is actually defined (by abuse of notation) in $H_{\mathrm{ét}}^4(X, \mathbb{Z}_{\ell}(2))$. Hence, c_2 defines a U -invariant class in $H_{\mathrm{ét}}^4(X_{\bar{s}}, \mathbb{Z}_{\ell}(2))$ for some open subgroup $U \subseteq \mathrm{Gal}(\bar{\mathbb{F}}_p, \mathbb{F}_p)$.

Now, it suffices to see that c_2 cannot be in the image of the cycle class map. We do not know of a source for the full ℓ -adic cohomology operations, which would say something about d_r for $r > 3$, so we argue as follows. If U is a smooth complex quasi-projective variety, then

$$H^i(U, \mathbb{Z}_{\ell}) \cong H_{\mathrm{ét}}^i(U, \mathbb{Z}_{\ell}), \tag{2}$$

where the first group is singular cohomology with coefficients in \mathbb{Z}_ℓ , and the second group is ℓ -adic cohomology, defined as usual by

$$H_{\text{ét}}^i(U, \mathbb{Z}_\ell) = \varprojlim_\nu H_{\text{ét}}^i(U, \mathbb{Z}/\ell^\nu).$$

In order to prove this, note that there are functorial comparison isomorphisms $H^i(U, \mathbb{Z}/\ell^\nu) \cong H_{\text{ét}}^i(U, \mathbb{Z}/\ell^\nu)$, where again the first term is singular cohomology with coefficients in \mathbb{Z}/ℓ^ν . Hence, to prove that (2) is an isomorphism, it suffices to show that the natural map

$$H^i(U, \mathbb{Z}_\ell) \rightarrow \varprojlim_\nu H^i(U, \mathbb{Z}/\ell^\nu)$$

is an isomorphism for any space U having the homotopy type of a finite CW complex. This is possible to prove by hand, but instead we refer to Bousfield and Kan [4, Theorem IX.3.1] for the Milnor exact sequence

$$0 \rightarrow \varprojlim_\nu^1 H^{i-1}(U, \mathbb{Z}/\ell^\nu) \rightarrow H^i(U, \mathbb{Z}_\ell) \rightarrow \varprojlim_\nu H^i(U, \mathbb{Z}/\ell^\nu) \rightarrow 0.$$

Since U has the homotopy type of a finite CW complex, the cohomology groups $H^{i-1}(U, \mathbb{Z}/\ell^\nu)$ are finitely generated, so the Mittag-Leffler condition is automatically satisfied, and the \varprojlim^1 term vanishes. So, we have proved that (2) is an isomorphism for smooth complex quasi-projective varieties U .

Let $K(U)[\beta^{-1}]_\ell$ denote the ℓ -adic completion of $K/\ell^\nu(U)[\beta^{-1}]$, the ℓ -adic Bott-inverted K -theory of Thomason [15, Appendix A]. Typically, the homotopy groups of this spectrum are written $K_*(U)[\beta^{-1}]_\ell$. If U is a quasi-projective scheme over \mathbb{C} , this is precisely the ℓ -complete (periodic) complex K -theory spectrum of U by [15, Theorem 4.11]: $K_m(U)[\beta^{-1}]_\ell \cong KU^{-m}(U)_\ell$. However, $K(U)[\beta^{-1}]$ is defined functorially over any base field, and there is a spectral sequence

$$E_2^{p,q} = H_{\text{ét}}^p(U, \mathbb{Z}_\ell(\frac{q}{2})) \Rightarrow K_{q-p}(U)[\beta^{-1}]_\ell,$$

which converges in good cases, for instance in the presence of uniform bounds on the ℓ -adic étale cohomological dimensions of the residue fields of U . The r th differential d_r has degree $(r, r-1)$. (Implicitly, if q is odd, the $E_2^{p,q}$ entry is zero.) See Thomason [15, Theorem 4.1]. If U is a \mathbb{C} -scheme, then the comparison between ℓ -adic Bott-inverted algebraic K -theory and periodic complex K -theory is compatible (via the Atiyah-Hirzebruch spectral sequence) with the comparison between singular and étale ℓ -adic cohomology. In particular, the differential d_r leaving the $E_r^{4,4}$ -term of the spectral sequence gives a cohomology operation on a subquotient of ℓ -adic cohomology which is compatible with base change.

In the Atiyah-Hirzebruch spectral sequence for $BG_{\mathbb{C}}$ we have that $d_r(c_2) \neq 0$. Using the compatibility established above, we see that $d_r(c_2) \neq 0$ in a subquotient of $H^{r+4}(X_{\overline{\mathbb{S}}}, \mathbb{Z}_\ell(2 + \frac{r-1}{2}))$. On the other hand, this operation d_r vanishes on the ℓ -adic cohomology of $BGL_{n, \overline{\mathbb{F}}_p}$, since r must be odd and $BGL_{n, \overline{\mathbb{F}}_p}$ does not support any odd cohomology. As $CH^2(X_{\overline{\mathbb{S}}})$ is generated by second Chern classes of representations, we see that c_2 cannot be in the image of the cycle class map. \square

4. Appendix: SAGE code

The following simple **SAGE** [14] script computes the second Chern classes, with help under the hood from **Singular** [6] for multivariable polynomials. The function `trim` serves to truncate polynomials to make the calculation faster by ignoring higher degree terms. The function `c2` returns n_p where $c_2(\gamma_n^p) = n_p c_2(\gamma_n)$ for a partition p . The input p is of the form (p_1, \dots, p_k) .

The main work in the code is done by the built-in SAGE function `SemistandardTableaux`, which returns all semistandard tableaux on a given partition. As explained in Fulton's book [8], these index a basis of γ^λ , which is then used to compute the Chern classes via the splitting principle. It seems easier to compute in the cohomology of classifying space of the maximal torus of GL_8 or GL_9 , so the code just cancels out any c_1 contributions. We do this by subtracting off kc_1^2 where k is the coefficient of x_1^2 in the resulting polynomial. The function then returns the coefficient of x_1x_2 .

```
def trim(f,n):
    g=0
    for m in f.monomials():
        if m.degree()<=n:
            g=g+f.monomial_coefficient(m)*m
    return g

def c2(n,p):
    R=PolynomialRing(ZZ,['x'+repr(i) for i in range(1,n+1)])
    q=1
    for t in SemistandardTableaux(p,max_entry=n).list():
        m=0
        for r in t:
            for c in r:
                m=m+eval('x'+repr(c))
                q=q*(1+m)
            q=trim(q,2)
    c1=(sum(v for v in R.gens()))^2
    r=q-q.monomial_coefficient(x1^2)*c1
```

This code will be maintained on the author's github page under the name `c2.sage` at gist.github.com/benjaminantieau and on the author's webpage.

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